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Anti-Periodic Boundary Conditions in Supersymmetric DLCQ

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Abstract

It is of considerable importance to have a numerical method for solving supersymmetric theories that can support a non-zero central charge. The central charge in supersymmetric theories is in general a boundary integral and therefore vanishes when one uses periodic boundary conditions. One is therefore prevented from studying BPS states in the standard supersymmetric formulation of DLCQ (SDLCQ). We present a novel formulation of SDLCQ where the fields satisfy anti-periodic boundary conditions. The Hamiltonian is written as the anti-commutator of two charges, as in SDLCQ. The anti-periodic SDLCQ we consider breaks supersymmetry at finite resolution, but requires no renormalization and becomes supersymmetric in the continuum limit. In principle, this method could be used to study BPS states. However, we find its convergence to be disappointingly slow.

1 Introduction

Supersymmetric gauge theories in low dimensions have been shown to be related to non-perturbative objects in M/string theory [1], and are therefore of particular interest nowadays. More dramatically, there is a growing body of evidence suggesting that gauged matrix models in $0+1$ and $1+1$ dimensions may offer a non-perturbative formulation of string theory [2, 3]. There is also a suggestion that large N gauge theories in various dimensions may be related to theories with gravity [6, 7, 8].

It is therefore interesting to study directly the non-perturbative properties of a class of supersymmetric matrix models at finite and large N_c, where N_c is the number of gauge colors. Supersymmetric Discrete Light Cone Quantization (SDLCQ)[12] is a unique non-perturbative numerical approximation that is manifestly supersymmetric at each stage of the calculation. In simplest terms SDLCQ is the approximation that arises when a theory is confined to a box of length L in the spatial light-cone directions. This leads to a discrete Fock space basis and the supercharges are finite-dimensional matrices in this basis. This combination of the DLCQ method and supersymmetry is well defined and has proven to be a powerful tool which has allowed to solve a large class of problem that have not been solved previously [7, 11, 14, 17]. It appears that supersymmetric theories are completely well defined, when formulated on a compact space [13, 15]. All of the models that have been addressed so far have had sufficient supersymmetry to make them completely finite so that no renormalization was necessary. It is important to note that numerical approximations that do break supersymmetry will still be faced with a renormalization problem.

In the light-cone formulation the supercharges Q^+_α and Q^-_α have several interesting and unique properties. Consider for example a pure Yang-Mills theory in D dimensions, which has a boson multiplet and a fermion multiplet. Since the longitudinal momentum operator is a kinematic operator, the supercharge Q^+_α must be quadratic in the fields, while the supercharge Q^-_α, whose square is the Hamiltonian, in general has both quadratic and cubic terms in the fields. The dynamics are carried by the cubic terms in the supercharge in the sense that a theory with only quadratic terms in Q^-_α will be a non-interacting theory. The supercharges contain an odd number of fermion fields therefore the fermion fields must be periodic[1].

We have formulated a number of supersymmetric theories imposing periodic boundary conditions on the fields and compared our results with other numerical results. We find excellent agreement and we find the SDLCQ converges extremely fast, much faster than

[1] We will not consider the possibility of twisted boundary conditions here, but clearly this is an interesting direction to explore.

any version of standard DLCQ [9, 10]. Among the most interesting results we found were a number of exactly massless bound states in some theories. These are states that are destroyed by one supercharge, Q^-_α, but not the other, Q^+_α, since they have finite momentum. These massless states persist at all values of the coupling and it is clear that these states are BPS states which saturate the bound $|Z| = M$ where the central charge Z is zero. It would be extremely interesting to find BPS states with non-zero masses numerically.

2 Formulation of the Theory

In a light-cone formulation the algebra of the supercharges with a central charge extension takes the form

$$\begin{aligned} \{Q^-_\alpha, Q^-_\beta\} &= P^-\delta_{\alpha,\beta} \\ \{Q^+_\alpha, Q^+_\beta\} &= P^+\delta_{\alpha,\beta} \\ \{Q^+_\alpha, Q^-_\beta\} &= P_\perp\gamma^\perp_{\alpha,\beta} + Z\delta_{\alpha,\beta}, \end{aligned} \tag{1}$$

where P^+ is the longitudinal momentum, $P_\perp$ is the transverse momentum and P^- is the Hamiltonian. Z is the central charge extension and we have suppressed the spinor indices. In light-cone quantized field theories one has transverse boost invariance so one can always work in a frame where the total transverse momentum is zero, thus $P_\perp$ on all physical states can be taken to be zero.

It is well known that the central charge extension Z can be written as an integral over the boundary of the space, and it will therefore vanish if one uses periodic boundary conditions. Therefore the last anti-commutator in Eq. (1) always takes the form

$$\{Q^+_\alpha, Q^-_\beta\} = 0. \tag{2}$$

Without a central charge extension the BPS states of a theory will simply be massless states. The mass of these states is protected by the BPS symmetry and they will remain massless at all couplings. We have seen these states in a recent SDLCQ calculation in 2+1 dimensions [4]. This theory is easily extended to an $\mathcal{N}$=2 supersymmetry, but without a central charge there is no hope of seeing massive BPS states. A massive BPS state would be a very striking feature in the spectrum the theory, since it would have a fixed mass as a function of the coupling.

We propose therefore to extend the definition of the supercharge in order to include anti-periodic boundary conditions. This extension breaks the supersymmetry at finite resolution, but at infinite resolution it will be restored. We define momentum-shifted analogues of the

standard supercharges, $Q^-_{\pm\frac{1}{2}}$, which carry momentum $\pm\frac{\pi}{2L}$. The Hamiltonian P^- is defined by

$$\{Q^-_{+\frac{1}{2},\alpha}, Q^-_{-\frac{1}{2},\beta}\} = 2\sqrt{2}P^-\delta_{\alpha,\beta}, \tag{3}$$

with the normalization of Ref. [12]. To test this method *in praxi*, we study the eigenvalue problem

$$2P^+P^-|\psi\rangle = M^2|\psi\rangle, \tag{4}$$

defined by the Hamiltonian, Eq. (3), within the context of the supersymmetric theory of adjoint fermions in $1+1$ dimensions. This is one of the simplest possible supersymmetric theories and all the low energy bound states of this theory are well known [16]. The supercharge density is $q^-(x)$

$$q^-(x) = ig2^{7/4}Tr\left[\psi(x)\psi(x)\frac{1}{\partial_-}\psi(x)\right]. \tag{5}$$

where $\psi(x)$ is a real adjoint fermion field. We expand the field into its modes which will carry half-integer momenta, because we impose anti-periodic boundary conditions

$$\psi(x)_{ij} = \sqrt{\frac{1}{4L}}\sum_{n=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots}\left(B_{ij}(n)e^{-in\pi x^-/L} + B^\dagger_{ji}(n)e^{in\pi x^-/L}\right). \tag{6}$$

We define $Q^-_{\pm\frac{1}{2}}$ to be

$$\begin{aligned} Q^-_{\pm\frac{1}{2}} &= \frac{1}{2L}\int_{-L}^{L} dx^- e^{\mp\pi x^-/2L} q^-(x) \qquad (7) \\ &= \frac{ig}{2^{1/4}}\sqrt{\frac{L}{\pi}}\sum_{n_1,n_2=\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots}\left(\frac{1}{n_1} - \frac{1}{n_2} - \frac{1}{n_1+n_2\mp\frac{1}{2}}\right)\times \\ &\times \left(B^\dagger_{ik}(n_1)B^\dagger_{kj}(n_2)B_{ij}(n_1+n_2\mp\frac{1}{2}) + B^\dagger_{ij}(n_1+n_2\pm\frac{1}{2})B_{ik}(n_1)B_{kj}(n_2)\right). \end{aligned}$$

Since we are using a real representation for the fermions it is easy to verify that the Fock space matrices $Q^-_{\pm\frac{1}{2}}$ satisfy,

$$Q^-_{+\frac{1}{2}} = \left[Q^-_{-\frac{1}{2}}\right]^t. \tag{8}$$

If all the wavefunctions of the physical states vanish sufficiently fast for small momentum, then for large resolution one would expect that the half unit of momentum which the supercharges carry is negligible compared to momentum the partons carry. Thus, $Q^-_{\pm\frac{1}{2}} \to Q^-$ in the continuum limit $K\to\infty$. However, since we have broken the supersymmetry one might face a mass renormalization which is generally required in these two-dimensional theories. In spite of this potential difficulty we proceed to calculate the spectrum of this theory, and will find that the theory indeed need not be renormalized.

3 Numerical Results

In order to obtain the mass squared eigenvalues M^2 of the theory, we have to solve the eigenvalue problem, Eq. (4). We use the standard large N_c discrete Fock basis. Its states are of the form

$$|p_1, p_2, \ldots, p_n\rangle = \frac{1}{{N_c}^{n/2}\sqrt{s}} Tr\left[B^\dagger(p_1)B^\dagger(p_2)\cdots B^\dagger(p_n)\right]|0\rangle,$$

where the symmetry factor s counts the number of times the set of momenta $(p_1, \ldots, p_n)$ is mapped onto itself under cyclic permutations. When using anti-periodic boundary conditions, the fermion bound states will lie in the sector of half-integer harmonic resolution K and boson bound states will have integer K. The action of the operators, Eq. (7), can be readily evaluated

$$\begin{aligned} Q^-_{\pm\frac{1}{2}}|p_1, p_2, \ldots, p_n\rangle = {} & \frac{ig\sqrt{L}}{2^{7/4}}\sqrt{\frac{N_c}{\pi}}\sum_{i=1}^{n}(-)^{i+1} \\ & \left\{(-)^{n(i+1)}\sum_{k=\frac{1}{2}}^{p_i\pm 1}\left(\frac{1}{k}+\frac{1}{p_i-k\pm\frac{1}{2}}-\frac{1}{p_i}\right)|k, p_i-k\pm\frac{1}{2}, L_n^{(i-1)}(p_i)\rangle\right. \\ & \left. +(-)^{ni}\left(\frac{1}{p_{i-1}}+\frac{1}{p_i}-\frac{1}{p_i+p_{i-1}\pm\frac{1}{2}}\right)|p_i+p_{i-1}\pm\frac{1}{2}, L_n^{(i-1)}(p_i, p_{i-1})\rangle\right\}, \end{aligned}$$

where $L_n^{(j)}$ is a permutation of $n-k$ momenta

$$L_n^{(j)}(p_{i_1}, \ldots, p_{i_k}) = \{p_{1+j}, p_{2+j}, \ldots, p_{i_1-1+j}, p_{i_1+1+j}, \ldots, p_{i_k-1+j}, p_{i_k+1+j}, \ldots p_{n+j}\}.$$

This compact form allows for an easy computer implementation. Since the supercharges change fermions to bosons and vice versa, they change the resolution K by $\pm\frac{1}{2}$. Thus in this discrete representation the supercharge matrix,

$$\langle K|Q^-_{\pm\frac{1}{2}}|K \mp \frac{1}{2}\rangle,$$

will in general not be a square matrix. The Hamiltonian, however, is constructed as the anti-commutator of the of the two momentum-shifted supercharge matrices, Eq. (3). From Eq. (8) it follows then that the Hamiltonian is a hermitian matrix. In this matrix representation the Hamiltonian at resolution K will receive contributions from intermediate states with resolution $K \pm \frac{1}{2}$. The supercharge matrices in a $(fermion, boson)^t$ basis have the structure

$$Q^-_{\pm\frac{1}{2}} = \begin{pmatrix} \mathbf{0} & A_{\pm\frac{1}{2}} \\ B_{\pm\frac{1}{2}} & \mathbf{0} \end{pmatrix} \tag{9}$$

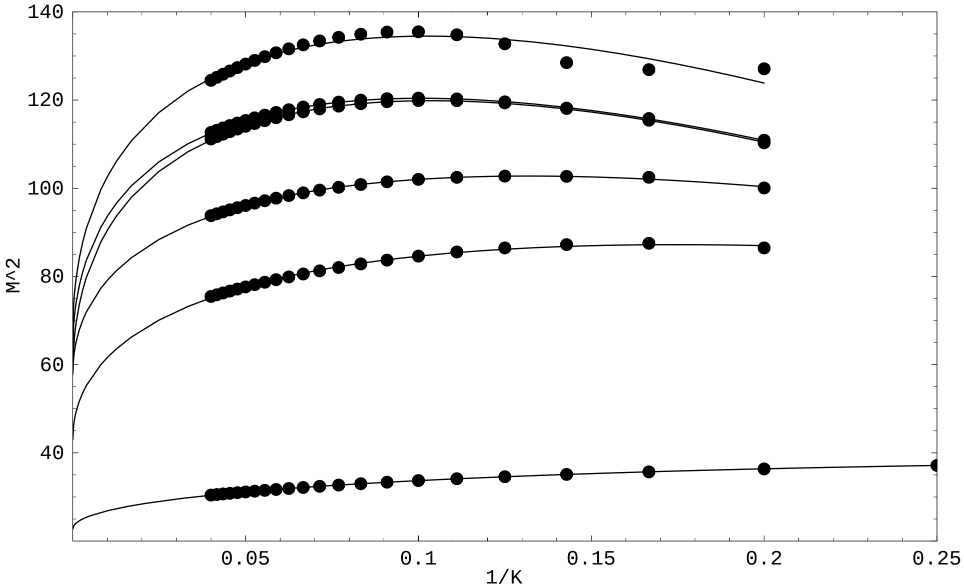

Figure 1: The six lowest masses in units of g^2N_c/π as a function of the inverse of the harmonic resolution $\frac{1}{K}$. All data were fitted to a function of the form $M^2 + a\sqrt{\frac{1}{K}} + b\frac{1}{K}$.

The Hamiltonian for the fermions is therefore,

$$P^-_{ferm} = A_{+\frac{1}{2}}B_{-\frac{1}{2}} + A_{-\frac{1}{2}}B_{+\frac{1}{2}}, \tag{10}$$

and the Hamiltonian for the bosons is,

$$P^-_{bose} = B_{+\frac{1}{2}}A_{-\frac{1}{2}} + B_{-\frac{1}{2}}A_{+\frac{1}{2}}. \tag{11}$$

These two matrices are subsequently diagonalized to yield the fermionic the bosonic masses. We note that these matrices have in general different dimensions.

In a first step, we calculated the fermionic and bosonic spectrum up to harmonic resolution K=10 by solving the eigenvalue problem, Eq. (4). We then focused on the six lowest eigenvalues whose eigenfunctions for resolutions up to K=10 are solely built out of states with at most four and five particles for the bosonic and fermionic spectrum, respectively. A truncation to four particles allowed us to go to resolution K=25 in the bosonic sector with relative ease. In the fermionic sector of the spectrum, the larger dimension of the fermionic matrix, Eq. (10), prevented us from going quite as high.

The resulting bound state masses in units of g^2N_c/π are shown in Fig. 1. We fitted all six curves to the function

$$M^2 + a\sqrt{\frac{1}{K}} + b\frac{1}{K}.$$

We find the continuum masses to be

$$M^2_{aSDLCQ} = 22.7, 43.0, 57.7, 58.0, 63.4, 67.0, \tag{12}$$

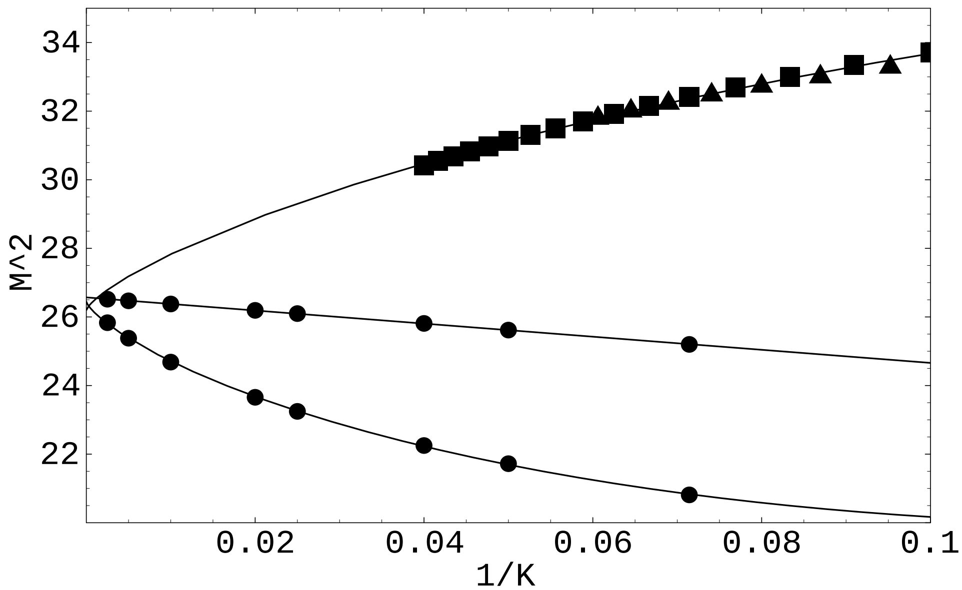

Figure 2: All the curves correspond to the lowest mass state (in units of $g^2 N_c/\pi$) of the theory. The boxes and triangles represent the lowest anti-symmetric SDLCQ boson and fermion eigenstates, respectively. The middle curve stems from a calculation using ordinary SDLCQ and the fermions and bosons are exactly degenerate. The lower curve is the standard DLCQ result for the lowest mass boson. The upper and lower curves are both fitted to $M^2 + a' K^{-3/4} + b' K^{-3/2}$.

compared to the values

$$M^2_{SDLCQ} = 23.8, 47.9, 62.1, 62.6, 63.8, 64.7, \tag{13}$$

of ordinary SDLCQ at resolution $K = 8$. While the agreement is far from perfect, it is clear that for this model the anti-periodic SDLCQ method works, in the sense that it correctly reproduces the spectrum of the theory. It is fair to say, however, that the exact continuum values are quite sensitive to the fitting function.

In Ref. [5] we compared the lowest eigenvalue as calculated in SDLCQ and standard DLCQ as a function of the resolution K. We found that the convergence of standard DLCQ is slow and clearly nonlinear as a function of $1/K$. Unfortunately, we see from Fig. 2 that the convergence for anti-periodic SDLCQ is, roughly speaking, about as bad as the standard Hamiltonian DLCQ method[2]. In Fig. 2 we show the lowest fermion bound state mass and the lowest boson bound state mass as a function of $1/K$ for the anti-periodic SDLCQ approximation. While the fermions and bosons appear at different resolutions, they can be fitted by the same curve for large enough K. Thus the supersymmetry of the

[2]For comparison, we used the same fitting function that was used in the DLCQ calculations of Ref. [5] for our data.

spectrum appears to be recovered by the approximation already at low harmonic resolution. Unfortunately, although the potential perils of supersymmetry breaking seem to be absent already at $K = 10$ for the lowest eigenstates, the convergence remains unpleasantly slow. In principle, however, the method appears to be working, and we anticipate room for numerical improvements.

4 Conclusions

In the present note we introduced a novel version of SDLCQ that allows for the use of anti-periodic boundary conditions for all fields. Consequently, the formalism can support for the first time boundary integral contributions. We provided evidence that the approximation converges to the standard supersymmetric results. The new approach inherits the very advantageous feature of absence of mass renormalization from SDLCQ. Unfortunately however, the approximation does not enjoy the same rapid convergence as SDLCQ. The lack of rapid convergence is a severe problem, since supersymmetric theories with non-trivial central charges ($\mathcal{N} \geq 2$) have many species of particles and therefore a very large Fock space. Thus going to sufficiently high resolution to obtain accurate results for the spectrum will be quite difficult. Of course, the standard DLCQ approach can be used with anti-periodic boundary conditions. But, as we saw in Fig. 2, its convergence is no better than that of the anti-periodic SDLCQ approximation. Moreover, the standard DLCQ approach will certainly have renormalization problems in higher dimensions.

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